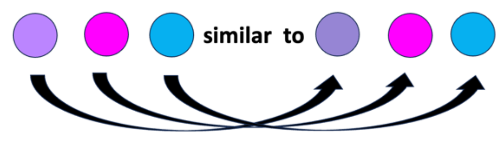
Similar Triangles

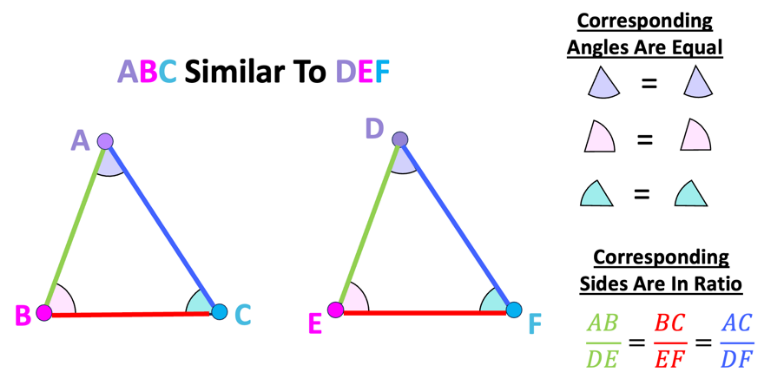
What is similarity?

Similar triangles are triangles that have the same shape, but their sizes may vary. In other words the only difference is size, not the angles. So, if the triangle was magnified or demagnified, it would superimpose the other triangle.

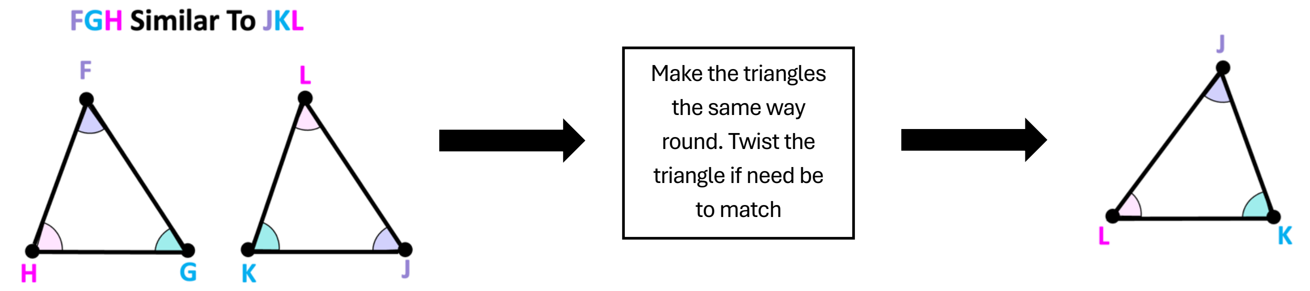
This is the main property of similar triangles: **All corresponding angles have the same size and corresponding sides are in the same ratio.**

ABC and DEF are similar means A corresponds to D , B corresponds to E and C corresponds to F

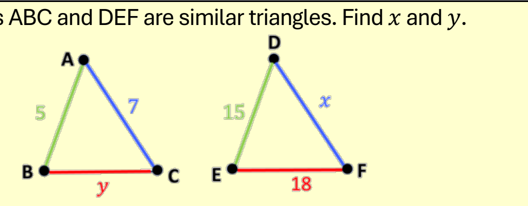
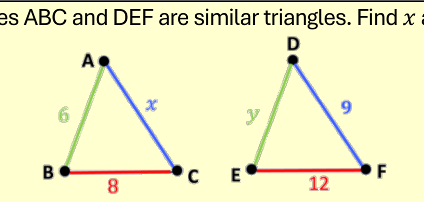




You will notice some similar triangles are flipped or rotated, but they are still called similar since the shape is same, just orientation differs.



Examples

Triangles ABC and DEF are similar triangles. Find x and y.	Triangles ABC and DEF are similar triangles. Find x and y.
	

We need to first find the ratio of the corresponding sides. From the green,

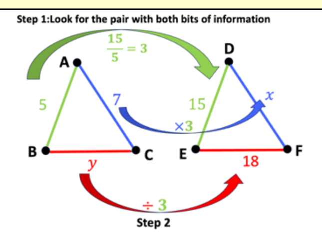
$$\frac{5}{15}=\frac{1}{3}$$

Hence, we can see the equal ratio implies

$$\frac{5}{15}=\frac{y}{18}=\frac{7}{x}$$

Hence,

$$y=6, \quad x=21$$



We need to first find the ratio of the corresponding sides. From the red,

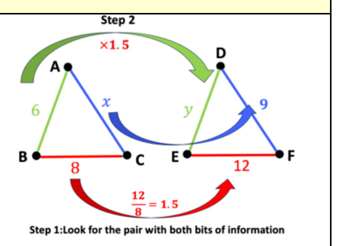
$$\frac{12}{8}=1.5$$

Hence, we can see the equal ratio implies

$$\frac{y}{6}=\frac{12}{8}=\frac{9}{x}$$

Hence,

$$y=9, \quad x=6$$

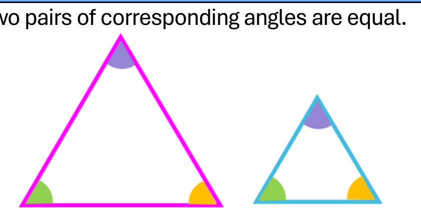


Criteria for Similarity

We use these criteria as the "test" for similarity. If triangles match any criterion, they are similar triangles. This lets us approach each problem in 3 steps:
1) Note each dimension in the triangle
2) Check for corresponding sides having same ratio and angles being equal
3) Apply criterion to establish similarity and use its properties

AA (Angle-Angle)

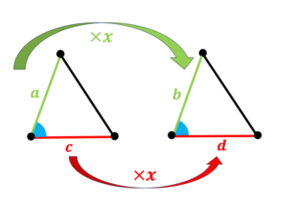
Two pairs of corresponding angles are equal.



Notice, we can pick any two angles, say purple and green being equal, and employ the AA criterion to call the triangles similar (since if two angles are the same, the third must be the same due to sum of triangles aways being 180)

SAS (Side-Angle-Side)

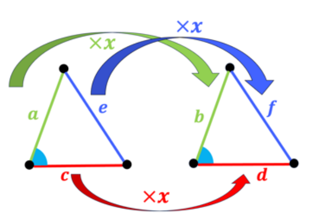
Ratio of **two pairs** of corresponding sides equal and their **included angle** is equal



As you see, the corresponding sides have the same ratio: $\frac{a}{b}=\frac{c}{d}=$. Also, the included angle is equal. Hence, we employ the SAS criterion to call the triangles similar.

SSS (Side-Side-Side)

Ratio of **all pairs** of corresponding sides equal

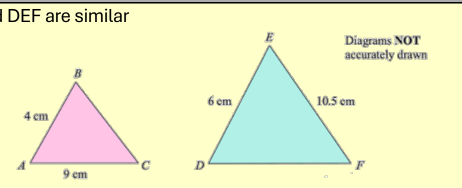


As you see, all the corresponding sides have the same ratio: $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=$. Hence, we employ the SSS criterion to call the triangles similar.

Easy Examples (Grade 5)

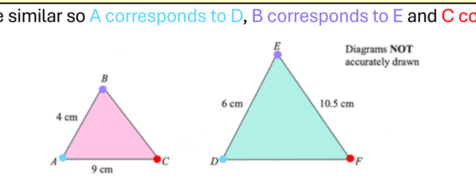
Example 1

Triangles ABC and DEF are similar



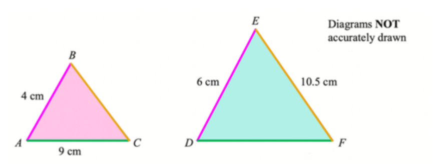
AB= 4 cm, AC= 9 cm, DE= 6 cm, EF = 10.5 cm
i. Work out the length of DF
ii. Work out the length of BC

ABC and DEF are similar so A corresponds to D, B corresponds to E and C corresponds to F.



The triangles are drawn the right way around since the corresponding points are in the same positions (i.e the coloured vertices below are in the same positions)

Let's label the corresponding sides with colour pairs



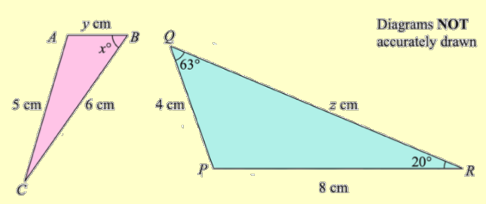
Way 1: Without algebra	Way 2: Use algebra
We pick a pair of known corresponding sides to find the length scale factor Length scale factor $=\frac{6}{4}=\frac{3}{2}$ Note: It is easier to always put the bigger number on top as done above and then decide whether we are getting bigger or smaller after and hence whether we should divide or multiply by the scale factor. DF corresponds to the side AC, so we can use length AC to help us find DF. DF is bigger than AC, so we multiply by the scale factor. $9\left(\frac{3}{2}\right)=\frac{27}{2}=13.5$	We can write the corresponding lengths (same colour pairs) as equivalent fractions $\frac{4}{6}=\frac{9}{DF}$ Now we can cross multiply $4DF=54$ $DF=13.5$ Note: There are many ways we could have built the equivalent fractions $\frac{6}{4}=\frac{DF}{9}$ or $\frac{4}{9}=\frac{6}{DF}$ or $\frac{9}{4}=\frac{DF}{6}$

ii.

Way 1: Without algebra	Way 2: Use algebra
We pick a pair of known corresponding sides to find the length scale factor Length scale factor $=\frac{6}{4}=\frac{3}{2}$ Note: It is easier to always put the bigger number on top as done above and then decide whether we are getting bigger or smaller after and hence whether we should divide or multiply by the scale factor EF corresponds to the side BC, so we can use length EF to help us find BC BC is smaller than EF so we divide by the scale factor $\frac{10.5}{\frac{3}{2}}=7$	We can write the corresponding lengths (same colour pairs) as equivalent fractions $\frac{4}{6}=\frac{BC}{10.5}$ Now we can cross multiply $6BC=42$ $DF=7$ Note: There are many ways we could have built the equivalent fractions $\frac{6}{4}=\frac{10.5}{BC}$ or $\frac{4}{BC}=\frac{6}{10.5}$ or $\frac{BC}{4}=\frac{10.5}{6}$

Example 2

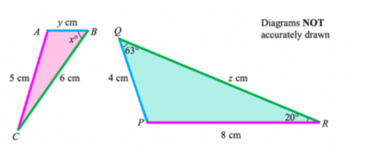
Here are two similar triangles



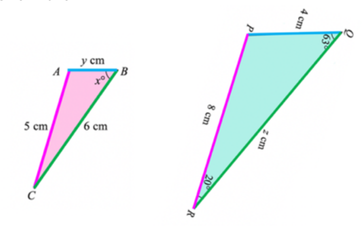
AB corresponds to PQ. BC corresponds to QR. Find the values of x, y, and z.

This question is a bit harder since the sides don't correspond in the order that the shapes are drawn

Let's label the corresponding sides with colour pairs to help.



Let's twist the shapes to match



Corresponding angles are always equal with similar shapes $x=63°$.
We now find y.

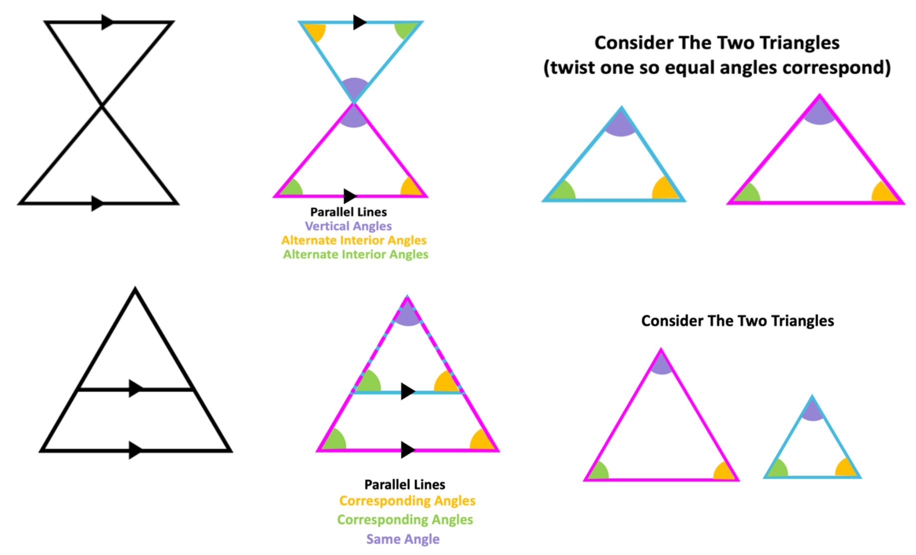
Way 1: Without algebra	Way 2: Use algebra
We pick a pair of known corresponding sides to find the length scale factor Length scale factor $=\frac{8}{5}$ Note: It is easier to always put the bigger number on top as done above and then decide whether we are getting bigger or smaller after and hence whether we should divide or multiply by the scale factor y corresponds to the side QP, so we can use length QP to help us find y y is smaller than QP so we divide by the scale factor $4\div\left(\frac{8}{5}\right)=2.5$	We can write the corresponding lengths as equivalent fractions $\frac{5}{8}=\frac{y}{4}$ Now we can cross multiply $8y=20$ $y=\frac{20}{8}=2.5$

We now find z.

Way 1: Without algebra	Way 2: Use algebra
We pick a pair of known corresponding sides to find the length scale factor Length scale factor $=\frac{8}{5}$ z corresponds to the side BC, so we can use length BC to help us find z z is bigger than BC so we multiply by the scale factor $6\left(\frac{8}{5}\right)=\frac{48}{5}=9.6$	We can write the corresponding lengths as equivalent fractions $\frac{5}{8}=\frac{6}{z}$ Now we can cross multiply $5z=48$ $z=\frac{48}{5}=9.6$

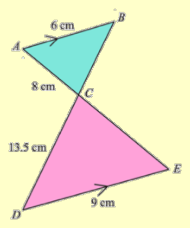
Medium Examples (grade 7)

Common diagrams with parallel lines

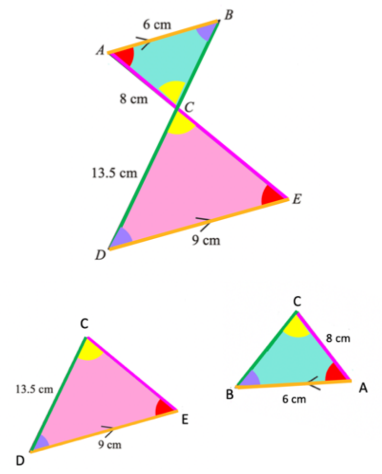


Example 1

AB is parallel to DE.



ACE and BCD are straight lines. AB = 6 cm, AC = 8 cm, CD = 13.5 cm, DE = 9 cm
i. Calculate the length of CE
ii. Calculate the length of BC



Now we proceed as usual
i.

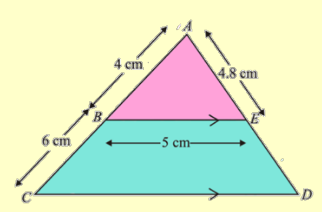
Way 1: Without algebra	Way 2: Use algebra
We pick a pair of known corresponding sides to find the length scale factor Length scale factor $=\frac{9}{6}=1.5$ CA corresponds to the side CE, so we can use length CA to help us find CE CE is larger than CA so we multiply by the scale factor $8(1.5)=12$	We can write the corresponding lengths as equivalent fractions $\frac{9}{6}=\frac{CE}{8}$ Now we can cross multiply $6CE=72$ $CE=12$

ii.

Way 1: Without algebra	Way 2: Use algebra
We pick a pair of known corresponding sides to find the length scale factor Length scale factor $=\frac{9}{6}=1.5$ BC corresponds to the side DC, so we can use length DC to help us find BC BC is smaller than DC so we divide by the scale factor $\frac{13.5}{1.5}=9$	We can write the corresponding lengths as equivalent fractions $\frac{9}{6}=\frac{13.5}{BC}$ Now we can cross multiply $9BC=81$ $BC=9$

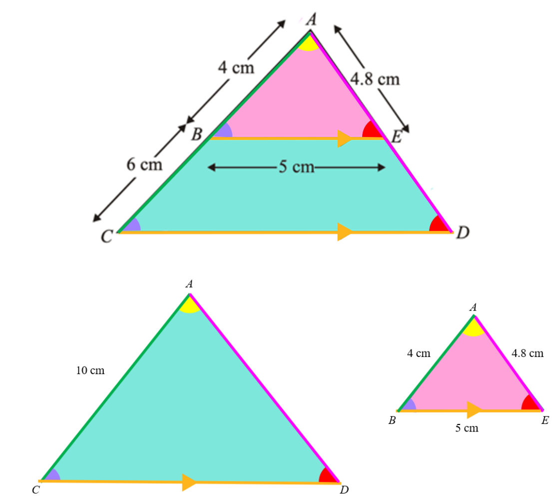
Example 2

BE is parallel to CD.



ABC and AED are straight lines.
AB = 4 cm, BC = 6 cm, BE = 5 cm, AE = 4.8 cm.
i. Calculate the length of CD.
ii. Calculate the length of ED.

The orange lines are parallel (given).



i.

Way 1: Without algebra	Way 2: Use algebra
We pick a pair of known corresponding sides to find the length scale factor. Length scale factor $=\frac{10}{4}=2.5$ BE corresponds to the side CD, so we can use length BE to help us find CD. CD is larger than BE, so we multiply by the scale factor. $5\times2.5=12.5$	We can write the corresponding lengths as equivalent fractions. $\frac{10}{4}=\frac{CD}{5}$ Now we can cross multiply. $4CD=50$ $CD=12.5$

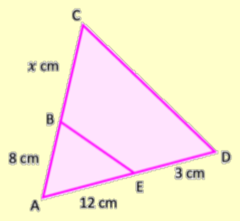
ii. In order to find ED we need to find AD and then subtract AE.

Way 1: Without algebra	Way 2: Use algebra
We pick a pair of known corresponding sides to find the length scale factor. Length scale factor $=\frac{10}{4}=2.5$ AD corresponds to the side AE, so we can use length AE to help us find AD. AD is larger than AE, so we multiply by the scale factor. $4.8\times2.5=12$ $ED=AD-AE$ $=12-4.8=7.2$	We can write the corresponding lengths as equivalent fractions. $\frac{10}{4}=\frac{AD}{4.8}$ Now we can cross multiply. $4AD=48$ $AD=12$ $ED=AD-AE$ $=12-4.8=7.2$

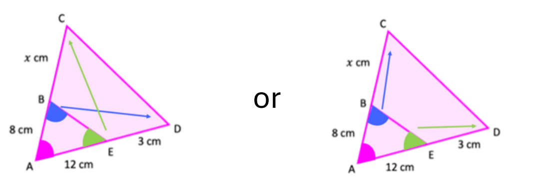
Somewhat Challenging Examples (Grade 8)

Example 1

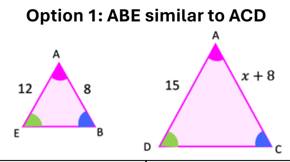
The two triangles in the diagram are similar. There are 2 possible values of x. Work out each of these values.



The question hasn't specified which two triangles are similar so it may be the reverse where ΔABE may be similar to ΔADC. We don't have parallel lines given like usual hence we can't say which angles correspond. So, we have 2 options for how these similar could look:

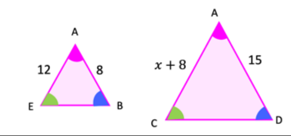


Option 1: ABE similar to ACD



Way 1: Without algebra	Way 2: Use algebra
Scale Factor $=\frac{15}{12}=1.25$ AC is larger than AB, so we multiply by the scale factor. $8\times1.25=10$ $AC=x+AB$ $10=x+8$ $x=2$	$\frac{12+3}{12}=\frac{8+x}{8}$ $8(12+3)=12(8+x)$ $120=96+12x$ $x=2$

Option 2 ABE similar to ADC

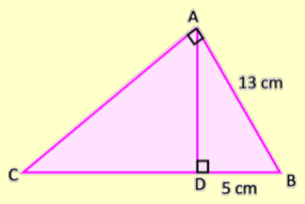


Way 1: Without algebra	Way 2: Use algebra
Scale Factor $=\frac{12+3}{8}=1.875$ AC is larger than AE, so we multiply by the scale factor. $12\times1.875=22.5$ $AC=x+AB$ $22.5=x+8$ $x=14.5$	We can write the corresponding lengths as equivalent fractions. $\frac{12+3}{8}=\frac{8+x}{12}$ $12(12+3)=8(8+x)$ $180=64+8x$ $x=14.5$

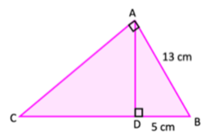
Example 2

ABC and ABD are two right angled triangles.

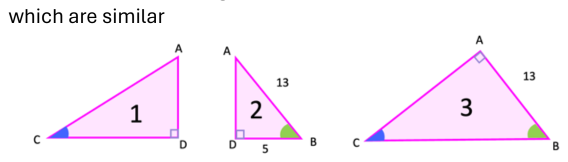
Angle BAC = Angle ADB = 90°



AB = 13 cm, DB= 5 cm
Work out the length of CB.

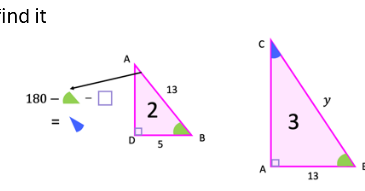


Let's indicate which angles ae the same first so we can see which are similar



1 and 3 are similar
2 and 3 are similar
They are similar since all the same angles are the same then the third is the same (AA criterion)

Let's look at 2 and 3 since we need to involve since CB since asked to find it

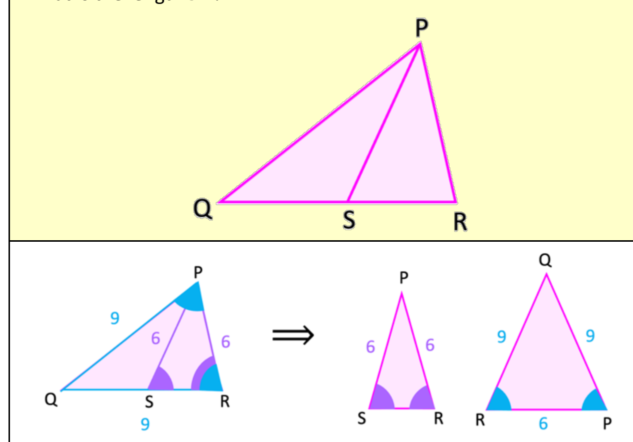


Way 1: Without algebra	Way 2: Use algebra
Use ratio of corresponding sides to find scale factor. Length scale factor $=\frac{13}{5}$ CB corresponds to the side AB, so we can use length AB to help us find CB. CB is larger than AB, so we multiply by the scale factor. $13\times\frac{13}{5}=33.8$	We can write the corresponding lengths as equivalent fractions. $\frac{13}{5}=\frac{CB}{13}$ Now we can cross multiply. $5CB=169$ $CB=33.8$

Harder Examples (Grade 8)

Example 1

PQR is a triangle and S is a point on QR.
$QP=QR$ = 9cm and $PR=PS$ = 6cm.
What is the length SR?



Since the purple and blue angles are the same at R, they must be equal. Hence, by AA criterion (2 angles the same), these are similar.

ΔPSR ≅ ΔQRP

So, using same ratio of corresponding sides gives

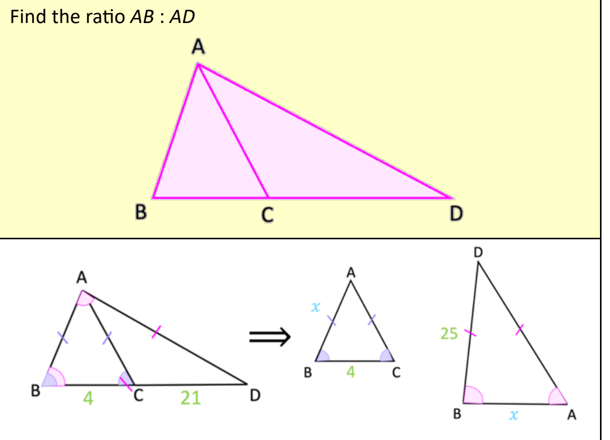
$$\frac{SR}{RP}=\frac{RP}{PQ}$$

$$\frac{SR}{6}=\frac{6}{9}$$

$$SR=4$$

Example 2

ABC and DAB are similar isosceles triangles.
$AB=AC$ and $AD=BD$
$BC:CD$ = 4:21
Find the ratio $AB:AD$



Since the purple and pink angles are the same at B, they must be equal. Hence, by AA criterion, these are similar.

ΔABC ≅ ΔDBA

So, using same ratio of corresponding sides gives

$$\frac{4}{x}=\frac{x}{25}$$

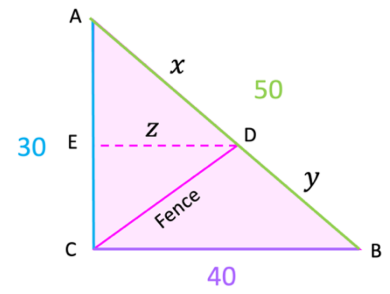
$$x^2=100$$
$$x=10$$

$$AB:AD=AB:BD=10:25=2:5$$

Example 3

A garden has the shape of a right-angled triangle with sides of length 30, 40 and 50. A straight fence goes from the corner with the right-angle to a point on the opposite side, dividing the garden into two sections which have the same perimeter. How long is the fence?

We can find the lengths of each individual piece on the triangles using the fact that the perimeter are the same and that AB = 50.



Same perimeter	Total length of AB is 50
$30+x+\text{fence}=40+y+\text{fence}$ $x-y=10$	$x+y=50$

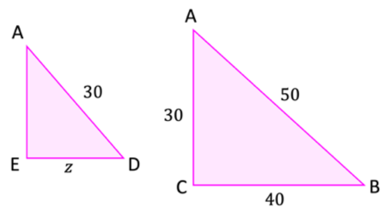
Solving simultaneously

$$x=30, \ y=20$$

If we can find ED and EC we can use Pythagoras

Since they share angle Angle EAD = Angle CAB and both have right angles, we can use AA criterion to call the triangles AED and ACB as similar.

$\frac{AE}{AC}=\frac{AD}{AB}\Rightarrow\frac{AE}{30}=\frac{30}{50}\Rightarrow AE=18$ Hence, we can find EC as $EC=30-18=12$	$\frac{ED}{CB}=\frac{AD}{AB}\Rightarrow\frac{ED}{40}=\frac{30}{50}\Rightarrow ED=24$



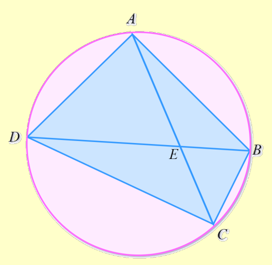
Now, we use Pythagorean Theorem as

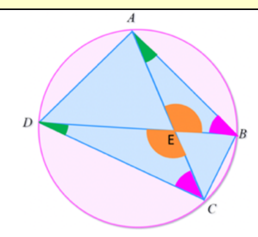
$$CD^2=EC^2+ED^2$$
$$CD^2=12^2+24^2$$
$$CD=12\sqrt{5}$$

Hardest Ever Examples – Combined With Congruent Shapes or Circles (Grade 9)

Example 1

A, B, C and D are four points on a circle. AEC and BED are straight lines. Prove the triangle ABE and DEC are similar





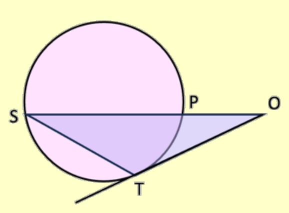
Angle AEB= Angle DEC (vertically opposite angles are equal)
Angle EDC=Angle EAB (angles in the same segment are equal)
Angle ABE=Angle DCE (angles in the same segment are equal)

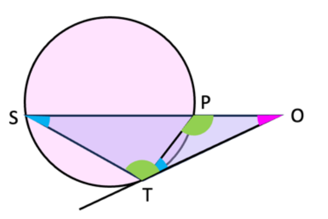
Hence, the triangles have 3 pairs of equal angles therefore similar by AA criterion.

Example 2

In the diagram, P, S and T are points on the circumference of a circle. O is the point such that
- OPS is a straight line
- OT is tangent to the circle

Prove that triangle OPT is similar to triangle OTS



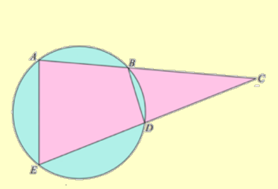


- O is same angle in both triangles, so $\angle POT=\angle TOS$
- Alternate segment theorem hence $\angle OTP=\angle TSO$

So, by AA criterion, triangle OPT is similar to triangle OTS.

Example 3

A, B, D and E are points on a circle
ABC and EDC are straight lines
Prove that triangle BCD is similar to triangle ECA

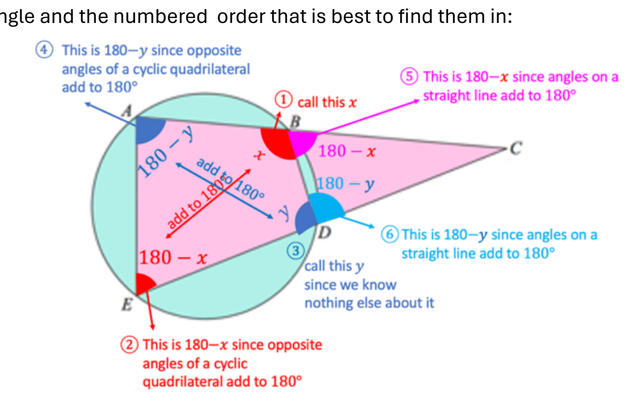


Here we are not given much information (not given any side lengths like usual). But we have a circle, so circle theorems should come to mind. We see a four-sided shape inside a circle which makes it likely that the cyclic quadrilateral theorem is applicable here.

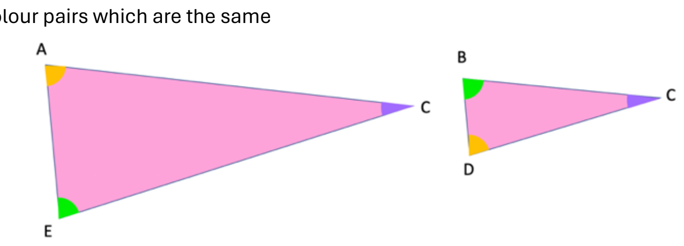
$\angle BCD=\angle ACE$ is common to both triangles which means we already know this angle is the same for both triangles.

Since we don't know any angles the easiest (and only) way to deal with this is to use algebra and start by calling one of the angles x

The diagram below shows the angle and the numbered order that is best to find them in:



So, we have the following colour pairs which are the same



$$\angle BCD=\angle BCD \text{ (common angle)}$$
$$\angle AEDC=\angle DBC \text{ (since we have shown both are equal to } 180-x)$$
$$\angle EAC=\angle BDC \text{ (since we have shown both are equal to } 180-y)$$

3 angles in both triangle ΔBCD and ΔECA are the same

Hence $\Delta BCD\cong\Delta ECA$